A geometric approach to correlations and quark number susceptibilities

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We study the thermodynamic geometry arising from the free energy for the 2- and 3-flavor finite temperature hot QCD near the critical temperature. We develop a geometric notion for QCD thermodynamics, relating it with the existing microscopic quantities, e.g. quark-number susceptibility, which appears naturally within an approximately self-consistent resummation of perturbative QCD. We further incorporate thermal fluctuations in the free energy, thus yielding the geometric properties of local and global chemical correlations. These investigations are perturbative in nature. Nevertheless, one could apply the same line of thought for the geometric realization of underlying quark susceptibilities, either in the fabric of lattice QCD or in that of non-perturbative QCD.

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I. INTRODUCTION

Thermodynamic geometry [1–3] has emerged as an important concept to understand the nature of correlations and physics of phase transition. It has been studied in the context of diverse thermodynamic systems: starting from the mixture of gases to the thermodynamics of black holes in string theory [4–6].

Quantum chromodynamics (QCD) on the other hand is a well celebrated physical theory of the strong interaction. QCD at high temperature (hot QCD) plays a crucial role in understanding the nature of the confinement-deconfinement phase transitions (hadronic phase to deconfined phase (quark-gluon plasma phase). The physics of strong interactions close to a phase transition point, T_c, is very complex and rich in this context.

Here, we have analyzed it in the important regime of the hot QCD by setting up firstly the thermodynamic geometric notion of the fluctuations of the hot QCD free energy, near the T_c and then investigated its connection with the existing studies of the quark-number susceptibilities [7–9]. The free-energy of QCD has been constructed employing the quasi-particle understanding of the 2- and 3-flavor hot QCD and the physical nature has been analyzed near the T_c. We have further incorporated the role of thermal fluctuations and thereby studied their impact from the notion of the thermodynamic geometry.

The main aim here is to investigate the relation between the covariant thermodynamic geometry and the quark-number susceptibility tensor. Both of these have recently received considerable attention in the black hole [4–6] and QCD thermodynamics [7–9].

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II. THERMODYNAMIC GEOMETRY

Let us now present a brief review of thermodynamic geometry from the perspective of the hot QCD. Let $F(\mu_i, T)$ be the free energy for given chemical potentials, and temperature $\{F, \mu_i, T\}$. Near the transition temperature, the free energy of hot QCD depends on the quark chemical potentials [13]. In this case, the T_C as the function of the quark chemical potentials, results in the following expression

$$\frac{T_c(\mu)}{T_c^0} = 1 + \sum_{i=1}^{2} \tilde{a}_i \frac{\mu_i^2}{(T_c^0)^2}, \qquad (1)$$

where, $\tilde{a}_1 = \tilde{a}_2 = -0.07$ for the 2-flavor case and $\tilde{a}_1 = \tilde{a}_2 = -0.114$ for the 3-flavor QCD. In general, considering the high temperature QCD as the effective quasi-particle system, we have constructed the thermodynamic geometric perspective of hot QCD [14] with the following free energy

$$F(\mu_1, \mu_2) := a_0(b + a_1\mu_1^2 + a_2\mu_2^2)^4 + (b + a_1\mu_1^2 + a_2\mu_2^2)^2$$
$$\times (c_1\mu_1^2 + c_2\mu_2^2) + c_4^{(1)}\mu_1^4 + c_4^{(2)}\mu_2^4, \qquad (2)$$

where $b = T_c^0$, $a_1 = \tilde{a}_1/b$ and $a_2 = \tilde{a}_2/b$. Here, the dimension of a_1, a_2 is in the unit of b^{-1}(GeV). The free energy is measured in units of GeV^4.

The free energy with the incorporation of the thermal fluctuations rectifies to the following expression

$$F(\mu_1, \mu_2) := a_0 + \frac{c_1\mu_1^2 + c_2\mu_2^2}{(b + a_1\mu_1^2 + a_2\mu_2^2)^2}$$
$$+ \frac{1}{2}\ln\left(\frac{c_1\mu_1^2 + c_2\mu_2^2}{(b + a_1\mu_1^2 + a_2\mu_2^2)^2}\right). \qquad (3)$$

Following [14], the chemical potentials μ_i are taken in the units of GeV. Thus, the components of the Weinhold metric tensor [1] are in the units of GeV^2 and the thermodynamic curvature is in GeV^4. In this study, the components of the metric tensor are,

$$g_{\mu_1\mu_1} = \frac{\partial^2 F}{\partial \mu_1^2}; \quad g_{\mu_1\mu_2} = \frac{\partial^2 F}{\partial \mu_1 \partial \mu_2}; \quad g_{\mu_2\mu_2} = \frac{\partial^2 F}{\partial \mu_2^2}. \qquad (4)$$

Subsequently, the determinant of the metric tensor and scalar curvature [14] are given by,

$$
\begin{aligned}
\|g\| &= F_{\mu_1\mu_1}F_{\mu_2\mu_2} - F_{\mu_1\mu_2}^2, \\
R &= -\frac{1}{2\|g\|^2}\Big(F_{\mu_2\mu_2}F_{\mu_1\mu_1\mu_1}F_{\mu_1\mu_2\mu_2} - F_{\mu_2\mu_2}F_{\mu_1\mu_1\mu_2}^2 \\
&\quad + F_{\mu_1\mu_2}F_{\mu_1\mu_1\mu_2}F_{\mu_1\mu_2\mu_2} + F_{\mu_1\mu_1}F_{\mu_1\mu_1\mu_2}F_{\mu_2\mu_2\mu_2} \\
&\quad - F_{\mu_1\mu_2}F_{\mu_1\mu_1\mu_1}F_{\mu_2\mu_2\mu_2} - F_{\mu_1\mu_1}F_{\mu_1\mu_2\mu_2}^2\Big), \quad (5)
\end{aligned}
$$

where the subscripts on F denote the respective partial derivatives. Interestingly, the thermodynamic curvature corresponds to the nature of the correlation present in the statistical system. For the two component systems [4], the scalar curvature can be identified as the correlation area $R(\mu_1,\mu_2) \sim \xi^2$, where $\sqrt{\xi(T_C)}$ defines the correlation length of the corresponding system, at a given temperature $T_C(\mu_1,\mu_2)$.

III. 2-FLAVOR QCD

Employing Eq.(4) and Eq.(5), we find that the determinant of the thermodynamic metric and the scalar curvature take the following forms:

$$
\begin{aligned}
g(\mu_1,\mu_2) &= \sum_{k,l=0|k+l\le6}^{6} a_{k,l}^{A}\mu_1^{2k}\mu_2^{2l}, \\
R(\mu_1,\mu_2) &= -\frac{4}{g^2}\sum_{k,l=0|k+l\le7}^{7} b_{k,l}^{A}\mu_1^{2k}\mu_2^{2l}. \quad (6)
\end{aligned}
$$

Their respective behavior is shown in the Fig. 1 for a given range of the interest $(\mu_1,\mu_2) \equiv (u,v)$.

Interestingly, the component of the metric tensor shows symmetry properties, *viz.* symmetry under the exchange of μ_1 and μ_2, which from the perspective of the quark susceptibility, turns out to be the parity symmetry. We find that the determinant of the metric tensor remains non-zero over the domain of chemical potentials and thus the thermodynamic configuration near T_C remains non-degenerate.

For the 2-flavor QCD, the inclusion of thermal fluctuation leads to the following positive definite determinant of the metric tensor and polynomial scalar curvature

$$
\begin{aligned}
g(\mu_1,\mu_2) &= 4\frac{\sum_{k,l=0|k+l\le7}^{7} a_{l,k}^{C}\ \mu_1^{2k}\mu_2^{2l}}{(\mu_1^2+\mu_2^2))^2(0.20-0.34(\mu_1^2-\mu_2^2))^7}, \\
R(\mu_1,\mu_2) &= -\frac{4}{g^2}(0.20-0.34(\mu_1^2+\mu_2^2) \\
&\quad \times \sum_{k,l=0|k+l\le12}^{12} a_{k,l}^{C}\ \mu_1^{2k}\mu_2^{2l}, \quad (7)
\end{aligned}
$$

where the coefficients $\{a_{i,j}\}$ appearing in the numerator of the curvature are depicted in the Fig.(2). It is ob-

served that thermal fluctuations preserve the symmetry properties of the thermodynamic geometry.

IV. 3-FLAVOR QCD

In this section, we discuss the thermodynamic geometry arising from the free energy of 3-flavor hot QCD. We obtain the following expression for the thermodynamic metric and the scalar curvature:

$$
\begin{aligned}
g(\mu_1,\mu_2) &= \sum_{k,l=0|k+l\le6}^{6} \tilde{a}_{k,l}^{A}\ \mu_1^{2l}\mu_2^{2k}, \\
R(\mu_1,\mu_2) &= -\frac{4}{g^2}\sum_{k,l=0|k+l\le7}^{7} \tilde{b}_{k,l}^{A}\ \mu_1^{2k}\mu_2^{2l} \quad (8)
\end{aligned}
$$

We find that the components of the metric tensor at zero chemical potentials take the values $g_{\mu_1,\mu_1} = 0.26$, $g_{\mu_2,\mu_2} = 0.22$ and $g_{\mu_1,\mu_2} = 0$, which in effect describe the diagonal and off diagonal quark susceptibility tensors. At zero chemical potentials, the determinant of the metric tensor reduces to $g = 0.06$ and the scalar curvature vanishes identically, which indicates a non-interacting statistical system.

For the 2-flavor QCD, considering the fact that the free energy is not symmetric under the exchange of the chemical potentials, we see for the 3-flavor QCD that $g(\mu_1,\mu_2) \ne g(\mu_2,\mu_1)$. Subsequently, the underlying diagonal quark susceptibility tensors differ even for vanishing chemical potentials. The same remains true for the determinant of the metric tensor and the associated scalar curvature. In fact, the inclusion of the quartic terms does not alter our observations. Finally, as for the 2-flavor QCD, we find that the symmetry of the metric and scalar curvature remains intact under the inclusion of thermal fluctuations, up to a refinement [14].

V. DISCUSSION AND CONCLUSION

In this paper, we have examined an intrinsic geometric notion to the 2-flavor and 3-flavor QCD thermodynamics. The free energy of the 2 and 3-flavor QCD, near T_c, with and without the logarithmic contributions under thermal fluctuations, introduces the thermodynamic geometric nature of the quark susceptibilities. Such a consideration is herewith shown to exhibit macroscopic versus microscopic duality relations via the fermion number density. These notions are anticipated to be realized as a mixing between the different flavors/colors, encoded in the corresponding partition function of the effective gauge field theory. It would be interesting to understand whether this kind of approach can be pushed further for QCD, for example in order to predict geometric properties of chemical correlations, under sizeable higher order perturbative and non-perturbative contributions.
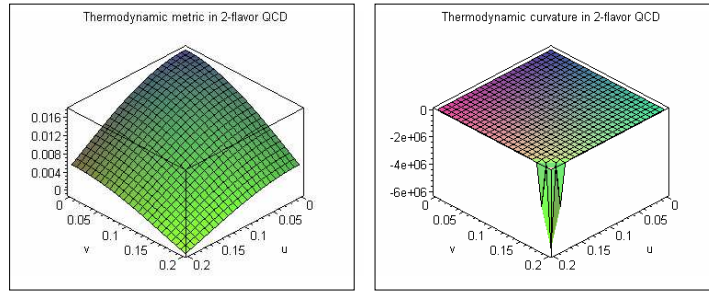
FIG. 1: The determinant of the thermodynamic metric and scalar curvature in 2-flavor QCD in the chemical potentials surface. Note that we measure the chemical potential in GeV.
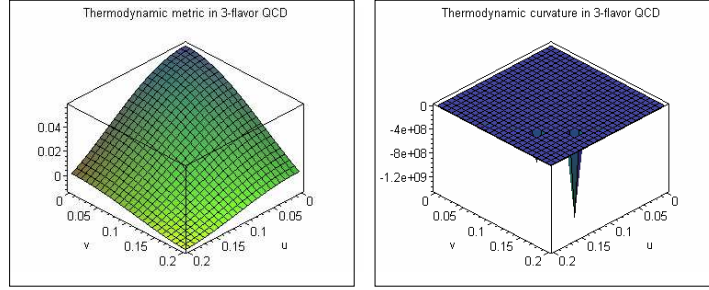

FIG. 2: The determinant of the thermodynamic metric and scalar curvature in 3-flavor QCD in the chemical potentials surface. Note that we measure the chemical potential in GeV.

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